Refined Brill-Noether Locus
and
Non-Abelian Zeta Functions for Elliptic Curves

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In this paper, new local and global non-abelian zeta functions for elliptic curves are defined using moduli spaces of semi-stable bundles. To understand them, we also introduce and study certain refined Brill-Noether locus in the moduli spaces. Examples of these new zeta functions and a justification of using only semi-stable bundles are given too. We end this paper with an appendix on the so-called Weierstrass groups for general curves, which is motivated by a construction of Euler systems using torsion points of elliptic curves.

1. Refined Brill-Noether Locus

1.1. Moduli Space of Semi-Stable Bundles

1.1.1 Indecomposable Bundles

Let E be an elliptic curve defined over $\overline{\mathbf{F}_q}$, an algebraic closure of the finite field $\mathbf{F}_p$ with q-elements.

Recall that a vector bundle V on E is called indecomposable if V is not the direct sum of two proper subbundles, and that every vector bundle on E may be written as a direct sum of indecomposable bundles, where the summands and their multiplicities are uniquely determined up to isomorphism. Thus to understand vector bundles, it suffices to study the indecomposable ones. To this end, we have the following result of Atiyah [At]. In the sequel, for simplicity, we always assume that the characteristic of $\mathbf{F}_q$ is strictly bigger than the rank of V.

Theorem. (Atiyah) *(a) For any $r \geq 1$, there is a unique indecomposable vector bundle I_r of rank r over E, all of whose Jordan-Hölder constituents are isomorphic to $\mathcal{O}_E$. Moreover, the bundle I_r has a canonical filtration*

$$\{0\} \subset F^1 \subset \ldots \subset F^r = I_r$$

with $F^i = I_i$ and $F^{i+1}/F^i = \mathcal{O}_E$;

(b) For any $r \geq 1$ and any integer a, relative prime to r and each line bundle λ over E of degree a, there exists up to isomorphism a unique indecomposable bundle $W_r(a;\lambda)$ over E of rank r with λ the determinant;

(c) The bundle $I_r(W_{r'}(a;\lambda)) = I_r \otimes W_{r'}(a;\lambda)$is indecomposable and every indecomposable bundle is isomorphic to $I_r(W_{r'}(a;\lambda))$ for a suitable choice of r, r', λ. Every bundle V over E is a direct sum of vector bundles of the form $I_{r_i}(W_{r_i'}(a_i;\lambda_i))$, for suitable choices of r_i, r_i' and λ_i. Moreover, the triples (r_i, r_i', λ_i) are uniquely specified up to permutation by the isomorphism type of V.

Here note in particular that $W_r(0,\lambda) \simeq \lambda$, and that indeed $I_r(W_{r'}(a;\lambda))$ is the unique indecomposable bundle of rank rr' such that all of whose successive quotients in the Jordan-Hölder filtration are isomorphic to $W_{r'}(a;\lambda)$.

1.1.2. Semi-Stable Bundles

As above, let V be a vector bundle over E. Define its slop $\mu(V)$ by $\mu(V) := \deg(V)/\mathrm{rank}(V)$. Then, following Mumford ([Mu]), V is called stable (resp. semi-stable), if for any proper subbundle W of V, $\mu(W) < \mu(V)$ (resp. $\mu(W) \leq \mu(V)$). For example, $I_r(W_{r'}(a;\lambda))$ is semi-stable with $\mu(I_r(W_{r'}(a;\lambda))) = a/r'$.

Theorem. (Atiyah) *(a) Every bundle V over E is isomorphic to a direct sum $\oplus_i V_i$ of semi-stable bundles, where $\mu(V_i) > \mu(V_{i+1})$;*

(b) Let V be a semi-stable bundle over E with slop $\mu(V) = a/r'$ where r' is a positive integer and a is an integer relatively prime to r'. Then V is a direct sum of bundles of the form $I_r(W_{r'}(a;\lambda))$, where λ is a line bundle of degree a.

1.1.3. Moduli Space of Semi-Stable Bundles

Let V be a semi-stable vector bundle, then we may associate it a Jordan-Hölder filtration, which is far from being unique. However the associated graded bundle, denoted as $\mathrm{gr}(V)$, is unique. Following Seshadri, two semi-stable vector bundles V and V' are called S-equivalent, denoted by $V \sim_S V'$, if $\mathrm{gr}(V) \simeq \mathrm{gr}(V')$.

Now set

$$\mathcal{M}_{E,r}(\lambda) = \{V : \mathrm{semi-stable}, \det V = \lambda, \mathrm{rank}(V) = r\}/\sim_S .$$

Then from the above classification we have the following well-known

Theorem. (Atiyah, Mumford-Seshadri) *With respect to a fixed pair (r, λ), there exists a natural projective algebraic variety structure on $\mathcal{M}_{E,r}(\lambda)$. Moreover, if $\lambda \in \mathrm{Pic}^0(E)$, then $\mathcal{M}_{E,r}(\lambda)$ is simply the projective space $\mathbf{P}^{r-1}_{\overline{\mathbf{F}}_q}$.*

1.2. Refined Brill-Noether Locus

1.2.1. Rational Points

Now let E be an elliptic curve defined over a finite field $\mathbf{F}_q$. Then over $\overline{E} = E \times_{\mathbf{F}_q} \overline{\mathbf{F}_q}$, from 1.1.3, we have the moduli spaces $\mathcal{M}_{\bar{E},r}(\lambda)$ (resp. $\mathcal{M}_{\bar{E},r}(d)$) of semi-stable bundles of rank r with determinant λ (resp. degree d) over $\bar{E}$. As algebraic varieties, we may consider $\mathbf{F}_q$-rational points of these moduli spaces. Clearly, by definition, these rational points of moduli spaces correspond exactly to these classes of semi-stable bundles which themselves are defined over $\mathbf{F}_q$. (In the case for $\mathcal{M}_{\bar{E},r}(\lambda)$, λ is assumed to be rational over $\mathbf{F}_q$.) Thus for simplicity, we simply write $\mathcal{M}_{E,r}(\lambda)$ or $\mathcal{M}_{E,r}(d)$ for the corresponding subsets of $\mathbf{F}_q$-rational points. For example, we then simply write $\mathrm{Pic}^0(E)$ for $\mathrm{Pic}^0(E)(\mathbf{F}_q)$. And by an abuse of notation, we often call these subsets the moduli spaces of semi-stable bundles too.

1.2.2. Standard Brill-Noether Locus

Note that if V is semi-stable with strictly positive degree d, then $h^0(E, V) = d$. Hence the standard Brill-Noether locus is either the whole space or empty. In this way, we are lead to study the case when $d = 0$.

For this, recall that for $\lambda \in \mathrm{Pic}^0(E)$,

$$\mathcal{M}_{E,r}(\lambda) = \{V : \mathrm{semi-stable}, \mathrm{rank}(V) = r, \det(V) = \lambda\}/\sim_S$$

is identified with

$$\{V = \oplus_{i=1}^r L_i : \otimes_i L_i = \lambda, L_i \in \mathrm{Pic}^0(E), i = 1, \dots, r\}/\sim_{\mathrm{iso}} \simeq \mathbf{P}^{r-1}$$

where $/\sim_{\mathrm{iso}}$ means modulo isomorphisms.

Now introduce the standard Brill-Noether locus

$$W_{E,r}^a(\lambda) := \{[V] \in \mathcal{M}_{E,r}(\lambda) : h^0(E, \mathrm{gr}(V)) \geq a\}$$

and its 'stratification' by

$$W_{E,r}^a(\lambda)^0 := \{[V] \in W_{E,r}(\lambda) : h^0(E, \mathrm{gr}(V)) = a\} = W_{E,r}^a(\lambda) \setminus \cup_{b \geq a+1} W_{E,r}^b(\lambda).$$

One checks easily that $W_{E,r}^a(\lambda) \simeq \mathbf{P}^{(r-a)-1}$. Thus in particular, we have the following

Lemma. *With the same notation as above,*

$$W_{E,r+1}^{a+1}(\lambda) \simeq W_{E,r}^a(\lambda), \qquad \text{and} \qquad W_{E,r+1}^{a+1}(\lambda)^0 \simeq W_{E,r}^a(\lambda)^0.$$

1.2.3. Refined Brill-Noether Locus

The Brill-Noether theory is based on the consideration of h^0. But in the case for elliptic curves, for arithmetic consideration, such a theory is not fine enough: not only h^0 plays important role, the automorphism groups are important as well. Based on this, we introduce, for a fixed $(k+1)$-tuple non-negative integers $(a_0; a_1, \dots, a_k)$, the subvariety of $W_{E,r}^{a_0}$ by setting

$$W_{E,r}^{a_0;a_1,\dots,a_k}(\lambda) := \{[V] \in W_{E,r}^{a_0}(\lambda) : \mathrm{gr}(V) = \mathcal{O}_E^{(a_0)} \oplus \oplus_{i=1}^k L_i^{(a_i)},\ \otimes_i L_i^{\otimes a_i} = \lambda,\ L_i \in \mathrm{Pic}^0(E),\ i = 1, \dots, k\}.$$

Moreover, we define the associated 'stratification' by setting

$$W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0 := \{[V]\in W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda),\ \#\{\mathcal{O}_E,L_1,\ldots,L_k\}=k+1\}.$$

From definition, we easily have the following

Lemma. *With the same notation as above,*

$$W_{E,r+1}^{a_0+1;a_1,\ldots,a_k}(\lambda)\simeq W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)$$

and

$$W_{E,r+1}^{a_0+1;a_1,\ldots,a_k}(\lambda)^0\simeq W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0.$$

Moreover,

$$\mathcal{M}_{E,r}(\lambda)=\cup_{a_0;a_1,\ldots,a_k}W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0,$$

where the union is a disjoint one.

In fact, the structures of $W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)$ can be given explicitly: They are products of (copies of) projective bundles over E and (copies of) projective spaces.

Proposition. *With the same notation as above, regroup $(a_0;a_1,\ldots,a_k)$ as $(a_0;b_1^{(s_1)},\ldots,b_l^{(s_l)})$ with the condition that $b_1>b_2>\ldots>b_l$ and $s_1,s_2,\ldots,s_l\in\mathbf{Z}_{>0}$, then*
(1) if $b_l=1$,

$$W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)\simeq\prod_{i=1}^{l-1}\mathbf{P}_E^{s_i-1}\times\mathbf{P}^{s_l};$$

(2) if $b_l>1$,

$$W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)\simeq\prod_{i=1}^{l}\mathbf{P}_E^{s_i}.$$

Proof. This is because we have the following two facts about the quitient of t products of elliptic curves:
(1) The quotient space $E^{(n)}/S_n$ is isomorphic to the $\mathbf{P}^{n-1}$-bundle over E; and
(2) The quotient of $E^{(n-1)}/S_n$ is isomorphic to $\mathbf{P}^{(n-1)}$. Here we embed $E^{(n-1)}$ as a subspace of $E^{(n)}$ under the map:

$$(x_1,\ldots,x_n)\mapsto(x_1,\ldots,x_{n-1},x_n)$$

with $x_n=\lambda-(x_1+x_2+\ldots+x_{n-1})$.

We end this subsection with the following intersection theoretical discussion. Fro simplicity, let $\lambda=\mathcal{O}_E$. Then from above, we have the refined Brill-Noether loci $W_{E,r}^{a_0;a_1,\ldots,a_k}(\mathcal{O}_E)$ which are isomorphic to products of (copies of) projective bundles over E and (copies of) projective spaces. Thus it would be very interesting to see the intersections of these special subvarieties in $\mathcal{M}_{E,r}(\mathcal{O}_E)=\mathbf{P}^{r-1}$. For this purpose, define the so-called Brill-Noether tautological ring $\mathbf{BN}_{E,r}(\mathcal{O}_E)$ to be the subring generated by all the associated refined Brill-Noether loci. For examples,
(1) If $r=2$, then this ring contains two elements: 1-dimensional one $W_{E,2}^{2;0}(\mathcal{O}_E)=\{[\mathcal{O}_E\oplus\mathcal{O}_E]\}$ and the whole $\mathbf{P}^1$;
(2) If $r=3$, then (generators of) this ring contains five elements: 2 of 0-dimensional objects: $W_{E,3}^{3;0}(\mathcal{O}_E)=\{[\mathcal{O}_E^{(3)}]\}$ and $W_{E,3}^{1;2}(\mathcal{O}_E)=\{[\mathcal{O}_E\oplus T_2^{(2)}]:T_2\in E_2\}$ containing 4 elements; 2 of 1-dimensional objects: $W_{E,3}^{1;1,1}=\{[\mathcal{O}_E\oplus L\oplus L^{-1}]:L\in\mathrm{Pic}^0(E)\}\simeq\mathbf{P}^1$, a degree 2 projective line contained in $\mathbf{P}^2=\mathcal{M}_{E,3}(\mathcal{O}_E)$; and $W_{E,3}^{0;2,1}=\{[L^{(2)}\oplus L^{-2}]:L\in\mathrm{Pic}^0(E)\}$ a degree 3 curve which is isomorphic to E; and finally the whole space. Moreover, the intersection of $W_{E,3}^{1;1,1}=\mathbf{P}^1$ and $W_{E,3}^{0;2,1}=E$ are supported on 0-dimensional locus $W_{E,3}^{1;1,1}$, with the multiplicity 3 on the single point locus $W_{E,3}^{3;0}(\mathcal{O}_E)$ and 1 on the completement of the points in $W_{E,3}^{1;1,1}$.

2. Invariants α, β and γ

2.1. Measure Refined Brill-Noether Locus Arithmetically

2.1.1. Invariant α

In the rest of this section, we use the same notation as in 1.2.

To measure the Brill-Noether locus, we introduce the following arithmetic invariant $\alpha_{E,r}(\lambda)$ by setting

$$\alpha_{E,r}(\lambda) := \sum_{V\in[V]\in\mathcal{M}_{E,r}(\lambda)} \frac{q^{h^0(E,V)}}{\#\mathrm{Aut}(V)}.$$

Also set

$$\alpha_{E,r}^{a_0+1;a_1,\ldots,a_k}(\lambda) := \sum_{V\in[V]\in W_{E,r}^{a_0+1;a_1,\ldots,a_k}(\lambda)^0} \frac{q^{h^0(E,V)}}{\#\mathrm{Aut}(V)}.$$

Before going further, we remark that above, we write $V\in[V]$ in the summation. This is because in each S-equivalence class $[V]$, there are usually more than one vector bundles V. For example, $[\mathcal{O}_E^{(4)}]$ consists of $\mathcal{O}_E^{(4)}$, $\mathcal{O}_E^{(2)}\oplus I_2$, $I_2\oplus I_2$, $\mathcal{O}_E\oplus I_3$, and I_4 by the result of Atiyah cited in 1.1.

Thus, by Lemma 1.2.3, we have the following

Lemma. *With the same notation as above,*

$$\alpha_{E,r}(\lambda) = \sum_{(a_0;a_1,\ldots,a_k);k} \alpha_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda).$$

We end this section with the following

Conjecture. *For all $\lambda\in\mathrm{Pic}^0(E)$,*

$$\alpha_{E,r}(\lambda) = \alpha_{E,r}(\mathcal{O}_E).$$

2.1.2. Invariants β and γ

Due to the importance of automorphism groups, following Harder-Narasimhan, and Desale-Ramanan, we introduce the following β-series invariants $\beta_{E,r}(d)$, $\beta_{E,r}(\lambda)$ and $\beta_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)$ by setting

$$\beta_{E,r}(d) := \sum_{V\in[V]\in\mathcal{M}_{E,r}(d)} \frac{1}{\#\mathrm{Aut}(V)},$$

$$\beta_{E,r}(\lambda) := \sum_{V\in[V]\in\mathcal{M}_{E,r}(\lambda)} \frac{1}{\#\mathrm{Aut}(V)},$$

and

$$\beta_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda) := \sum_{V\in[V]\in W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0} \frac{1}{\#\mathrm{Aut}(V)}.$$

Corresponding to the Conjecture 2.1.1 for α, for β, we have the following deep

Theorem. ([HN] & [DR]) *For all $\lambda,\lambda'\in\mathrm{Pic}^d(E)$,*

$$\beta_{E,r}(\lambda) = \beta_{E,r}(\lambda').$$

Moreover,

$$N_1\cdot\beta_{E,r}(\lambda) = \frac{N_1}{q-1}\cdot\prod_{i=2}^r \zeta_E(i) - \sum_{\Sigma_1^k r_i=r,\Sigma_i d_i=d,\frac{d_1}{r_1}>\ldots>\frac{d_k}{r_k},k\geq 2} \prod_i \beta_{E,r_i}(d_i)\frac{1}{q^{\Sigma_{i<j}(r_jd_i-r_id_j)}}.$$

Here N_1 denotes $\#E(:=\#E(\mathbf{F}_q))$ and $\zeta_E(s)$ denotes the Artin zeta function for elliptic curve $E/\mathbf{F}_q$.

Thus, we are lead to introduce the γ-series invariants $\gamma_{E,r}(\lambda)$ and $\gamma_{E,r}^{a_0+1;a_1,\ldots,a_k}(\lambda)$ by setting

$$\gamma:=\alpha-\beta.$$

That is to say,

$$\gamma_{E,r}(\lambda):=\sum_{V\in[V]\in\mathcal{M}_{E,r}(\lambda)}\frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)},$$

and

$$\gamma_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda):=\sum_{V\in[V]\in W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0}\frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)}.$$

Clearly, by the above Theorem, the Conjecture 2.1.2 for α is equivalent to the following

Conjecture. *For all $\lambda\in\mathrm{Pic}^0(E)$,*

$$\gamma_{E,r}(\lambda)=\gamma_{E,r}(\mathcal{O}_E).$$

The advantage of this Conjecture is that now the support of the summation is over $W_{E,r}^1(\lambda)$, a codimension 1 projective subspace.

Similarly, we have the following

Lemma. *With the same notation as above, for $\lambda\in\mathrm{Pic}^0(E)$,*

$$\beta_{E,r}(\lambda)=\sum_{a_0;a_1,\ldots,a_k}\beta_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)$$

and

$$\gamma_{E,r}(\lambda)=\sum_{a_0;a_1,\ldots,a_k}\gamma_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda).$$

2.2. Relations Between β and γ

2.2.1. Bundles with Trivial Graded Bundles

After certain painful calculations, by an accident, we are lead to the following

Conjecture. *For all $\lambda\in\mathrm{Pic}^0(E)$,*

$$\gamma_{E,r+1}^{a_0+1;a_1,\ldots,a_k}(\lambda)=\beta_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda).$$

Note that by Lemma 1.2.3,

$$W_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)^0\simeq W_{E,r+1}^{a_0+1;a_1,\ldots,a_k}(\lambda)^0.$$

Hence, from the definition, to verify the latest Conjecture, it suffices to show that for any fixed $(L_1,\ldots,L_k)\in\big(\mathrm{Pic}^0(E)\big)^{(k)}$ such that $\#\{\mathcal{O}_E,L_1,\ldots,L_k\}=k+1$, we have

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(a_0)}\oplus\oplus_{i=1}^kL_i^{(a_i)}}\frac{1}{\#\mathrm{Aut}(V)}=\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(a_0+1)}\oplus\oplus_{i=1}^kL_i^{(a_i)}}\frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)}.$$

Thus by looking at the structure of the automorphism groups carefully from the condition that $\mathcal{O}_E,L_1,\ldots,L_k$ are all different, we see that this latest conjecture is equivalent to the following

Main Conjecture (in Algebraic Theory of Non-Abelian Zeta Functions for Elliptic Curves.) *For any $r\in\mathbf{Z}_{>0}$,*

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(r)}}\frac{1}{\#\mathrm{Aut}(V)}=\sum_{W:\mathrm{gr}(W)=\mathcal{O}_E^{(r+1)}}\frac{q^{h^0(E,W)}-1}{\#\mathrm{Aut}(W)}.$$

Clearly, the advantage of the Main Conjecture is that only trivial bundle and its extensions are involved.

Before going further, to convince the reader, we check the example with $r=3$. In this case, there are three possibilities for V. That is, $\mathcal{O}_E^{(3)}$, $\mathcal{O}_E \oplus I_2$ and I_3. One checks the cardinal numbers of the corresponding automorphism groups are $(q^3-1)(q^3-q)(q^3-q^2)$, $(q-1)^2q^3$ and $(q-1)q^2$ respectively. Similarly, there are five possibilities for W. That is, $\mathcal{O}_E^{(4)}$, $\mathcal{O}_E^{(2)} \oplus I_2$, $I_2^{(2)}$, $\mathcal{O}_E \oplus I_3$ and I_4. The cardinal numbers of the corresponding automorphism groups are $(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)$, $(q-1)q^3(q^3-q)(q^3-q^2)$, $q^4(q^2-1)(q^2-q)$, $(q-1)^2q^4$ and $(q-1)q^3$, and their h^0 are given by $4,3,2,2,1$ respectively. Therefore, the Main Conjecture is equivalent to, in case $r=3$, the following relation:

$$\begin{aligned}
&\frac{1}{(q^3-1)(q^3-q)(q^3-q^2)}+\frac{1}{(q-1)^2q^3}+\frac{1}{(q-1)q^2}\\
&\qquad=\frac{q^4-1}{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}+\frac{q^3-1}{(q-1)q^3(q^3-q)(q^3-q^2)}\\
&\qquad\quad+\frac{q^2-1}{q^4(q^2-1)(q^2-q)}+\frac{q^2-1}{(q-1)^2q^4}+\frac{q-1}{(q-1)q^3}.
\end{aligned}$$

We leave the routine check of this latest identity to the reader, who may certainly be amused if doing correctly.

2.2.2. Main Conjecture Implies All Conjectures

It suffices to imply Conjecture 2.1.2 from the Main Conjecture. For this, by Lemma 2.1.2, we need to show that

$$\sum_{a_0;a_1,\ldots,a_k}\gamma_{E,r}^{a_0;a_1,\ldots,a_k}(\lambda)=\sum_{a_0;a_1,\ldots,a_k}\gamma_{E,r}^{a_0;a_1,\ldots,a_k}(\mathcal{O}_E).$$

Now by the Main Conjecture, which is equivalent to Conjecture 2.2.1, the left hand side becomes

$$\sum_{a_0;a_1,\ldots,a_k}\beta_{E,r-1}^{a_0-1;a_1,\ldots,a_k}(\lambda)$$

while the right hand side is simply

$$\sum_{a_0;a_1,\ldots,a_k}\beta_{E,r-1}^{a_0-1;a_1,\ldots,a_k}(\mathcal{O}_E).$$

Note also that $\gamma_{E,r}^{0;a_1,\ldots,a_k}(\lambda)=0$. Therefore the left hand side is simply $\beta_{E,r-1}(\lambda)$ while the right hand side becomes $\beta_{E,r-1}(\mathcal{O}_E)$. Thus by Theorem 2.1.2, we complete the proof of the following

Theorem. *Assume that for any $1\leq r'\leq r$,*

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(r')}}\frac{1}{\#\mathrm{Aut}(V)}=\sum_{W:\mathrm{gr}(W)=\mathcal{O}_E^{(r'+1)}}\frac{q^{h^0(E,W)}-1}{\#\mathrm{Aut}(W)}.$$

Then

$$\alpha_{E,r}(\lambda)=\beta_{E,r-1}(\mathcal{O}_E)$$

for all $\lambda\in\mathrm{Pic}^0(E)$. In particular,

$$\alpha_{E,r}(0)=N_1\cdot\alpha_{E,r}(\mathcal{O}_E),\qquad \beta_{E,r}(0)=N_1\cdot\beta_{E,r}(\mathcal{O}_E),\qquad\text{and}\quad \gamma_{E,r}(0)=N_1\cdot\gamma_{E,r}(\mathcal{O}_E).$$

Thus, theoretically, with the relation $\alpha-\beta=\gamma$, we then can determine all invariants $\alpha_{E,r},\beta_{E,r}$ and $\gamma_{E,r}$. (We reminder the reader that in practice the precise formula in Theorem 2.1.2 is hardly useful as there are too many infinite summations involved.)

3. New Non-Abelian Zeta Functions for Elliptic Curves

3.1. Non-Abelian Local Zeta Functions

3.1.1. Definition

Let E be an elliptic curve defined over $\mathbf{F}_q$, the finite field with q elements. Then we have the associated ($\mathbf{F}_q$-rational points of) moduli spaces $\mathcal{M}_{E,r}(d)$. By definition, the *rank r non-abelian zeta function $\zeta_{E,r,\mathbf{F}_q}(s)$ of E* is defined by setting

$$\zeta_{E,r,\mathbf{F}_q}(s) := \sum_{V\in[V]\in\mathcal{M}_{E,r}(d),d\geq 0} \frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)} q^{-s\cdot d(V)}, \qquad \mathrm{Re}(s)>1.$$

Here $d(V)$ denotes the degree of V.

Remark. We call the above infinite sum the rank r non-abelian zeta function because when $r=1$ the above summation $\zeta_{E,1}(s)$ coincides with the classical Artin zeta function for the elliptic curve E over $\mathbf{F}_q$. In other words, the classical Artin zeta function $\zeta_E(s)$ may be written as

$$\zeta_E(s) = \sum_{L\in\mathrm{Pic}^d(E),d\geq 0} \frac{q^{h^0(E,L)}-1}{\#\mathrm{Aut}(L)} q^{-s\cdot d(L)}, \qquad \mathrm{Re}(s)>1,$$

where $d(L)$ denotes the degree of L.

3.1.2. Basic Properties

With the above definition, by a direct yet long calculation, we have, by setting $t=q^{-s}$ and $Z_{E,r,\mathbf{F}_q}(t) := \zeta_{E,r,\mathbf{F}_q}(s)$, the following

Fundamental Identity. *Let E be an elliptic curve defined over $\mathbf{F}_q$. Then for any $r\in\mathbf{Z}_{>0}$,*

$$Z_{E,r,\mathbf{F}_q}(t) := \gamma_{E,r}(0)\cdot\frac{P_{E,r,\mathbf{F}_q}(t)}{(1-t^r)(1-q^rt^r)}.$$

Here

$$\begin{aligned}&.P_{E,r,\mathbf{F}_q}(t)\\ =&1+\sum_{i=1}^{r-1}(q^i-1)\frac{\beta_{E,r}(i)}{\gamma_{E,r}(0)}\cdot t^i+\Big((q^r-1)\frac{\beta(0)}{\gamma_{E,r}(0)}-(q^r+1)\Big)\cdot t^r+\sum_{i=1}^{r-1}(q^i-1)\frac{\beta_{E,r}(i)}{\gamma_{E,r}(0)}\cdot q^{r-i}t^{2r-i}+q^rt^{2r}.\end{aligned}$$

Remark. We in this paper choose not to write down the detailed elementary calculations, despite the fact that some of them are very long and sometimes a bit complicated.

As a direct consequence of this Fundamental Identity, we have the following

Theorem. *Let E be an elliptic curve defined over $\mathbf{F}_q$, the finite field with q elements. Then the associated rank r non-abelian zeta function $\zeta_{E,r,\mathbf{F}_q}(s)$ satisfies the following basic properties:*

*(1) (***Rationality***) $Z_{E,r,\mathbf{F}_q}(t)$ may be written as the quotient of two polynomials;*

*(2) (***Functional Equation***) $\zeta_{E,r,\mathbf{F}_q}(s)=\zeta_{E,r,\mathbf{F}_q}(1-s)$.*

From here we may conclude that after suitable arrangement, the product of two reciprocal roots of $P_{E,r,\mathbf{F}_q}(t)$, a degree $2r$ polynomial, are always equal to q, the cardinal number of the base field. Moreover, we know that up to the term $\gamma_{E,r}(0)$, from Theorem 2.1.2, the coefficients of these local non-abelian zeta functions can be computed. Thus if the Main Conjecture is assumed, then all terms of our non-abelian zeta functions can be given precisely.

Moreover, note that the moduli spaces $\mathcal{M}_{E,r}(d)$ are indeed projective bundles over E. Thus if the main conjecture holds, in the study of non-abelian zeta functions for elliptic curves, we are lead to study only the vertical direction, i.e., the Brill-Noether loci appeared a single fiber (and hence all fibers). So, the horizontal direction, i.e., the Picard group, plays no role. This is why we call the elegant conjectural relation

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(r)}}\frac{1}{\#\mathrm{Aut}(V)} = \sum_{W:\mathrm{gr}(W)=\mathcal{O}_E^{(r+1)}}\frac{q^{h^0(E,W)}-1}{\#\mathrm{Aut}(W)}$$

the Main Conjecture in algebraic theory of non-abelian zeta functions for elliptic curves.

3.2. Non-Abelian Global Zeta Functions

3.2.1. Definition.

Now let $\mathbf{E}$ be an elliptic curve defined over a number field F. For a fixed positive integer r, set S_{bad} to be the union of all infinite places (of F), all finite places where $\mathbf{E}$ have bad reductions, or where the characteristics of the residue fields are less than r. By definition, a (finite) places v of F is good, if v is not in S_{bad}.

Thus, in particular, for good places v, by taking reduction of $\mathbf{E}$ at v, we have the associated regular elliptic curve $\mathbf{E}_v$ defined over $F(v) \simeq \mathbf{F}_{q_v}$, the residue field of F at v, where q_v denotes cardinal number of $F(v)$. Then by applying the construction of 3.1, we get the rank r local non-abelian zeta functions $\zeta_{\mathbf{E}_v,r,\mathbf{F}_{q_v}}(s)$. In particular, we further obtain, by the rationality, the corresponding polynomials $P_{\mathbf{E}_v,r,\mathbf{F}_q}(t)$ of degree $2r$ (with 1 as constant terms).

Definition. *Let $\mathbf{E}$ be an elliptic curve defined over a number field F. Then for any positive integer r, define its associated rank r global non-abelian zeta function $\zeta_{\mathbf{E},r,F}(s)$ by setting*

$$\zeta_{\mathbf{E},r,F}(s) := \prod_{v:\text{good}} P_{\mathbf{E}_v,r,\mathbf{F}_{q_v}}(q_v^{-s})^{-1}.$$

Here q_v denotes the cardinal number of the residue field of F at v.

Clearly, if $r = 1$, this then recovers the famous Hasse-Weil zeta function for elliptic curves, for which we have the celebrated BSD conjecture. Thus it seems to be quite natural for us to call the above Euler product a global non-abelian zeta function for elliptic curve.

Surely the biggest problem we are now facing in this algebraic part of our non-abelian zeta function is to give the precise region over which the Euler product in the definition converges. For this we have to use our refined Brill-Noether theory discussed in Section 1.

3.2.2. Estimations for β and γ

We now want to give estimations for invariants β and γ. So let E be an elliptic curve defined over a finite field $\mathbf{F}_q$ as before.

First we study $\beta_{E,r}$. For this, following Harder-Narasimhan, we interpret the Tamagawa number of $\mathrm{SL}(n)$ is 1 as follows:

Proposition. (=[DR Proposition 1.1]) *Let $\zeta_E(s)$ be the Artin zeta function of E. Then for any fixed $\lambda \in \mathrm{Pic}^d(E)$,*

$$\sum_{V:\mathrm{rank}(V)=r,\det(V)=\lambda} \frac{1}{\#\mathrm{Aut}(V)} = \frac{1}{q-1}\prod_{k=2}^{r}\zeta_E(k).$$

Therefore, for a fixed $\lambda \in \mathrm{Pic}^d(E)$,

$$\sum_{V:\mathrm{rank}(V)=r,\det(V)=\lambda} \frac{1}{\#\mathrm{Aut}(V)} = O(q^{-1}).$$

This implies in particular that

$$\beta_{E,r}(\lambda) = O(q^{-1}).$$

Thus, by Hasse's result ([Ha]) on $N_1 = \#\mathrm{Pic}^d(E)$, we know $N_1 = O(q)$. Therefore we complete the following

Proposition I. *With the same notation as above, $\beta_{E,r}(d) = O(1)$, as $q \to \infty$.*

Next we study $\gamma_{E,r}(0)$. As our final purpose is to give a good estimation for the coefficients of $P_{E,r,\mathbf{F}_q}(t)$, hence by the Fundamental Identity in 3.1.2, it suffices to give a lower bound for $\gamma_{E,r}(0)$. For this, we consider semi-stable vector bundles V with $\mathrm{gr}(V) = \mathcal{O}_E \oplus \oplus_{i=1}^{r-1} L_i$ with $L_i \in \mathrm{Pic}^0(E)$ and $\#\{\mathcal{O}_E, L_1, \ldots, L_r\} = r$. Clearly then $V = \mathrm{gr}(V)$ and there are totally $O(N_1^{r-1})$ or better $O(q^{r-1})$ of them, as $q \to \infty$ by the above mentioned result of Hasse. On the other hand, easily, we have $h^0(V) = 1$ and $\#\mathrm{Aut}(V) = (q-1)^r$. Therefore

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E\oplus\oplus_{i=1}^{r-1},L_i\in\mathrm{Pic}^0(E),\#\{\mathcal{O}_E,L_1,\ldots,L_r\}=r+1} \frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)} = O(1).$$

This then implies the following

Proposition II. *With the same notation as above, $\gamma_{E,r}(d) = O(1)$, as $q \to \infty$.*

Remark. In fact if we assume the Main Conjecture in 2.2.1, i.e.,

$$\sum_{V:\mathrm{gr}(V)=\mathcal{O}_E^{(r)}} \frac{1}{\#\mathrm{Aut}(V)} = \sum_{W:\mathrm{gr}(W)=\mathcal{O}_E^{(r+1)}} \frac{q^{h^0(E,W)}-1}{\#\mathrm{Aut}(W)},$$

then by Theorem 2.2.2, $\gamma_{E,r}(0) = \beta_{E,r-1}(0)$. Hence Proposition II is a direct consequence of Proposition I.

3.2.3. Convergence of Global Non-Abelian Zeta Functions

As above, let E be an elliptic curve defined over $\mathbf{F}_q$. Then by the Fundamental Identity and the Functional Equation for local non-abelian zeta functions of elliptic curves,

$$P_{E,r,\mathbf{F}_q}(t) = \prod_{i=1}^r (1 + A_i t + qt^2)$$

with $A_i \in \mathbf{R}$. Therefore, we have

$$\sum_i A_i = (q-1)\frac{\beta_{E,r}(1)}{\gamma_{E,r}(0)}; \qquad \sum_{i<j} A_iA_j = (q^2-1)\frac{\beta_{E,r}(2)}{\gamma_{E,r}(0)};$$

$$\ldots, \qquad \prod_{i=1}^r A_i = (q^r-1)\frac{\beta_{E,r}(0)}{\gamma_{E,r}(0)} - (q^r+1).$$

So, as $q \to \infty$, by Proposition I and II in 3.2.1,

$$\begin{aligned} \sum_i A_i =& O(q); \\ \sum_{i<j} A_iA_j =& O(q^2); \\ &\ldots\ldots\ldots\ldots\ldots \\ \prod_{i=1}^r A_i =& O(q^r). \end{aligned}$$

So, $A_i = O(q), i = 1, \ldots, r$. As a direct consequence, we then have the following

Theorem. *Let $\mathbf{E}$ be an elliptic curve defined over a number field F. Then all rank r global non-abelian zeta function $\zeta_{\mathbf{E},r,F}(s)$, defined by (infinite) Euler products converge when* $\mathrm{Re}(s) > 2$.

Remark. When $r = 1$, by the above mentioned result of Hasse, the Hasse-Weil zeta functions, i.e., $\zeta_{\mathbf{E},1,F}(s)$ in our notation, converge when $\mathrm{Re}(s) > \frac{3}{2}$. While this result is better than ours in the case when $r = 1$, we are also quite satisfied with our one as our Theorem is the best possible for general r at this stage. (See e.g. 4.1.1 below.)

Now, a natural question is whether our rank r global non-abelian zeta functions have meromorphic extensions and satisfy the functional equation. We believe that it should be the case. In fact we have the following

Working Hypothesis. *By introducing also factors for bad places, the completed rank r non-abelian zeta functions $\xi_{\mathbf{E},r,F}(s)$ for elliptic curves have meromorphic continuation to the whole complex plane and satisfy the functional equation*

$$\xi_{\mathbf{E},r,F}(s) = \xi_{\mathbf{E},r,F}(1 + \frac{1}{r} - s).$$

Clearly, we then would also hope for certain type of such zeta functions, the inverse Mellin transform would lead to modular forms of fractional weight $1+\frac{1}{r}$. Unfortunately, we have not yet obtained any examples

to support this speculation. But if it holds, we then have a systematical way to construct fractional weight modular forms, for which, except in the case of half integers we know very little.

4. Examples and Justifications

4.1. Lower Rank Non-Abelian Zeta Functions

4.1.1. Rank Two

Let E be an elliptic curve defined over the finite field $\mathbf{F}_q$. If rank r is two, we need then only to calculate $\beta_{E,2}(0), \beta_{E,2}(1)$ and $\gamma_{E,r}(0)$.

We first consider $\beta_{E,2}(0)$. Then by our discussion on Brill-Noether locus, it suffices to calculate $\beta_{E,2}(\mathcal{O}_E)$. Now

$$\mathcal{M}_{E,2}(\mathcal{O}_E) = W_{E,2}^{2;0}(\mathcal{O}_E)^0 \cup W_{E,2}^{0;2}(\mathcal{O}_E)^0 \cup W_{E,2}^{0;1,1}(\mathcal{O}_E)^0.$$

Clearly,

$$W_{E,2}^{2;0}(\mathcal{O}_E)^0 = \{[V] : \mathrm{gr}(V) = \mathcal{O}_E^{(2)}\}$$

consisting of just 1 element;

$$W_{E,2}^{0;2}(\mathcal{O}_E)^0 = \{[V] : \mathrm{gr}(V) = T_2^{(2)}, T_2 \in E_2, T_2 \neq \mathcal{O}_E\}$$

consisting of 3 elements coming from non-trivial $T_2 \in E_2$, 2-torsion subgroup of E; while

$$W_{E,2}^{0;1,1}(\mathcal{O}_E)^0 = \{[V] : \mathrm{gr}(V) = L \oplus L^{-1}, L \in \mathrm{Pic}^0(E), L \neq L^{-1}\}$$

is simply the complement of the above 4 points in $\mathbf{P}^1$. With this, one checks that

$$\beta_{E,2}(0) = \Big(\frac{1}{(q^2-1)(q^2-q)} + \frac{1}{(q-1)q}\Big) + 3\cdot\Big(\frac{1}{(q^2-1)(q^2-q)} + \frac{1}{(q-1)q}\Big) + \big(q+1-(3+1)\big)\cdot\frac{1}{(q-1)^2} = \frac{q+3}{q^2-1}.$$

And hence

$$\beta_{E,2}(0) = N_1 \cdot \frac{q+3}{q^2-1}.$$

As for $\beta_{E,2}(1)$, it is very simple: Any degree one rank two semi-stable bundle is stable. Moreover, by the result of Atiyah cited in 1.1, there is exactly one stable rank two bundle whose determinant is the fixed line bundle. Thus

$$\beta_{E,2}(1) = N_1 \cdot \frac{1}{q-1}.$$

Finally, we study $\gamma_{E,2}(0)$. We want to check Conjecture 2.1.2. Clearly if $\lambda \neq \mathcal{O}_E$, then $\gamma_{E,2}(\lambda)$ is supported on

$$W_{E,2}^{1;1}(\lambda) = \{[V] : \mathrm{gr}(V) = \mathcal{O}_E \oplus \lambda\}$$

consisting only one element with $V = \mathrm{gr}(V) = \mathcal{O}_E \oplus \lambda$. So

$$\gamma_{E,2}(\lambda) = \frac{q-1}{(q-1)^2} = \frac{1}{q-1}.$$

On the other hand, $\gamma_{E,2}(\mathcal{O}_E)$ is supported on

$$W_{E,2}^{2;0}(\mathcal{O}_E) = \{[V] : \mathrm{gr}(V) = \mathcal{O}_E^{(2)}\}$$

consisting only one element too. However, now in the single class $[V]$ with $\mathrm{gr}(V) = \mathcal{O}_E^{(2)}$, there are two elements, i.e., $\mathcal{O}_E^{(2)}$ and I_2. So

$$\gamma_{E,2}(\mathcal{O}_E) = \frac{q^2-1}{(q^2-1)(q^2-q)} + \frac{q-1}{(q-1)q} = \frac{1}{q-1} = \beta_{E,1}(\mathcal{O}_E).$$

Thus we have checked all conjectures in the case $r=2$.

Proposition. *With the same notation as above, in the case $r=2$, all conjectures in this paper are confirmed. In particular,*

$$Z_{E,2,\mathbf{F}_q}(t)=\frac{N_1}{q-1}\cdot\frac{1+(q-1)t+(2q-4)t^2+(q^2-q)t^3+q^2t^4}{(1-t^2)(1-q^2t^2)}.$$

We reminder the reader that in this case, $P_{E,2,\mathbf{F}_q}(t)$ is independent of E and with integer coefficients. Thus for global rank two non-abelian zeta function, we obtain an absolute Euler product, say in the case when the base field is $\mathbf{Q}$,

$$E_2(s)=\prod_{p\geq 3,\mathrm{prime}}\frac{1}{1+(p-1)p^{-s}+(2p-4)p^{-2s}+(p^2-p)p^{-3s}+p^2p^{-4s}},\qquad \mathrm{Re}(s)>2.$$

Clearly we expect more from such a beautiful Euler product.

4.1.2. Rank Three

First, we check whether the Main Conjecture in 2.1.2 holds. That is to say, we should show

$$\sum_{V,\mathrm{gr}(V)=\mathcal{O}_E^{(2)}}\frac{1}{\#\mathrm{Aut}(V)}=\sum_{W,\mathrm{gr}(W)=\mathcal{O}_E^{(3)}}\frac{q^{h^0(E,V)}-1}{\#\mathrm{Aut}(V)}.$$

By the fact that $\mathrm{Aut}(\mathcal{O}_E\oplus I_2)=(q-1)^2q^3$, the above identity is equivalent to

$$\frac{1}{(q^2-1)(q^2-q)}+\frac{1}{(q-1)q}=\frac{q^3-1}{(q^3-1)(q^3-q)(q^3-q^2)}+\frac{q^2-1}{(q-1)^2q^3}+\frac{q-1}{(q-1)q^2},$$

which may be directly checked. This together with the similar relation for $r=1$ discussed in 4.1.1 leads to

$$\gamma_{E,3}(0)=N_1\cdot\gamma_{E,3}(\mathcal{O}_E)=N_1\cdot\beta_{E,2}(\mathcal{O}_E)=N_1\cdot\frac{q+3}{q^2-1}.$$

So we are left to study $\beta_{E,3}(d)$, $d=0,1,2$. Easily, we have

$$\beta_{E,3}(1)=\beta_{E,3}(2)=N_1\cdot\frac{1}{q-1}$$

since here all semi-stable bundles become stable. Thus we are lead to consider only $\beta_{E,3}(0)$. So it suffices to give $\beta_{E,3}(\lambda)$ for any $\lambda\neq\mathcal{O}_E$. (Despite the fact that $\beta_{E,r}(\lambda)=\beta_{E,r}(\mathcal{O}_E)$ for any $\lambda\in\mathrm{Pic}^0(E)$, in practice, the calculation of $\beta_{E,r}(\lambda)$ with $\lambda\neq\mathcal{O}_E$ is easier than that for $\beta_{E,r}(\mathcal{O}_E)$.)

Now

$$\mathcal{M}_{E,3}(\lambda)=\Big((W_{E,3}^{2;1}(\lambda)^0)\cup W_{E,3}^{1;2}(\lambda)^0\cup W_{E,3}^{1;1,1}(\lambda)^0\Big)\cup W_{E,3}^{0;3}(\lambda)^0\cup W_{E,3}^{0;2,1}(\lambda)^0\cup W_{E,3}^{0;1,1,1}(\lambda)^0.$$

Moreover, we have

(1) $W_{E,3}^{2;1}(\lambda)^0$ consists a single class [V], i.e., the one with $\mathrm{gr}(V)=\mathcal{O}_E^2\oplus\lambda$, which contains two vector bundles, i.e., $\mathcal{O}_E^2\oplus\lambda$ and $I_2\oplus\lambda$;

(2) $W_{E,3}^{2;1}(\lambda)^0\cup W_{E,3}^{1;2}(\lambda)^0\cup W_{E,3}^{1;1,1}(\lambda)^0\simeq\mathbf{P}^1$ with $W^{1;2}(\lambda)^0$ consists of 4 classes $[V]$, i.e., these such that $\mathrm{gr}(V)=\mathcal{O}_E\oplus\big(\lambda^{\frac12}\big)^{(2)}$, where $\lambda^{\frac12}$ denotes any of the four square roots of λ. Clearly then in each class $[V]$, there are also two vector bundles $\mathcal{O}_E\oplus\big(\lambda^{\frac12}\big)^{(2)}$ and $\mathcal{O}_E\oplus I_2\otimes\lambda^{\frac12}$;

(3) $W_{E,3}^{0;3}(\lambda)^0\cup W_{E,3}^{0;2,1}(\lambda)^0\cup W_{E,3}^{0;1,1,1}(\lambda)^0=\mathbf{P}^2\backslash\mathbf{P}^1$.

(3.a) $W_{E,3}^{0;3}(\lambda)^0$ consists of 9 classes $[V]$, i.e., these $[V]$ with $\mathrm{gr}(V)=\big(\lambda^{\frac13}\big)^{(3)}$ where $\lambda^{\frac13}$ denotes any of the 9 triple roots of λ. Moreover, in each $[V]$, there are three bundles, i.e., $\big(\lambda^{\frac13}\big)^{(3)}$, $\lambda^{\frac13}\oplus I_2\otimes\lambda^{\frac13}$ and $I_3\otimes\lambda^{\frac13}$.

(3.b) $\Big(W_{E,3}^{2;1}(\lambda)^0\cup W_{E,3}^{1;2}(\lambda)^0\Big)\cup\Big(W_{E,3}^{0;3}(\lambda)^0\cup W_{E,3}^{0;2,1}(\lambda)^0\Big)$ is isomorphic to E. Moreover, each class $[V]$ in $W^{0;2,1}(\lambda)^0$ consists of two bundles, i.e., $L^{(2)}\oplus\lambda\otimes L^{-2}$ and $I_2\otimes L\oplus\lambda\otimes L^{-2}$ when $\mathrm{gr}(V)=L^{(2)}\oplus\lambda\otimes L^{-2}$.

(One checks that in fact the refined Brill-Noether loci $\mathbf{P}^1$ and E appeared above are embedded in $\mathbf{P}^2$ as degree 2 and 3 regular curves. And hence the intersection should be 6: The intersection points are at $[V]$ with $\mathrm{gr}(V)=\mathcal{O}_E^{(2)}\oplus\lambda$ with multiplicity 2, and $\mathcal{O}_E\oplus\big(\lambda^{\frac12}\big)^{(2)}$ corresponding to four square roots of λ with multiplicity one. That is to say, the intersection actually are supported on $W_{E,3}^{2;1}(\lambda)^0\cup W_{E,3}^{1;2}(\lambda)^0$. So it would be very interesting in general to study the intersections of the refined Brill-Noether loci as well.)

From this analysis, we conclude that

$$\begin{aligned}\beta_{E,3}(\lambda)=&\Big(\frac{1}{(q^2-1)(q^2-q)(q-1)}+\frac{1}{(q-1)q(q-1)}\Big)\\&+4\Big(\frac{1}{(q^2-1)(q^2-q)(q-1)}+\frac{1}{(q-1)q(q-1)}\Big)+(q-4)\cdot\Big(\frac{1}{(q-1)^3}\Big)\\&+9\Big(\frac{1}{(q^3-1)(q^3-q)(q^3-q^2)}+\frac{1}{(q-1)^2q^3}+\frac{1}{(q-1)q^2}\Big)\\&+\big(N_1-(9+4+1)\big)\cdot\Big(\frac{1}{(q-1)(q^2-1)(q^2-q)}+\frac{1}{(q-1)(q-1)q}\Big)\\&+\big(q^2-(N_1-4-1)\big)\cdot\frac{1}{(q-1)^3}.\end{aligned}$$

Clearly, it is not of our best interests to write down the associated zeta function very precisely. Still, in this case, all conjectures in this paper have been confirmed. In particular, we conclude that $r=2$ is the only case that the global zeta functions are independent of elliptic curves and that the polynomials are with integral coefficients instead of rational coefficients in general.

Also we would reminder the reader that here in fact 2- and 3- torsion points are involved naturally in the calculation.

4.2. Why Use only Semi-Stable Bundles

4.2.1. Degree 0

At the first glance, Theorem 2.1.2 and Proposition 3.2.2 suggest that in the definition of non-abelian zeta functions we should consider all vector bundles, just as what happens in the theory of automorphic L-functions. However, we here use an example with $r=2$ to indicate the opposite.

Thus we frist introduce a new zeta function $\zeta_{E,r}^{\mathrm{all}}(s)$ by

$$\zeta_{E,r}^{\mathrm{all}}(s):=\sum_{V:\mathrm{rank}(V)=2}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}\cdot q^{-sd(V)}.$$

Then by our discussion on the non-abelian zeta functions associated to semi-stable bundle, we only need to consider the contribution of rank 2 bundles which are not semi-stable.

We start with a discussion on extension of bundles. Assume that V is not semi-stable of rank 2. Let L_2 be the line subbundle of V with maximal degree, then V is obtained from the extension of $L_2:=V/L_1$ by L_1

$$0\to L_1\to V\to L_2\to 0.$$

But V is not semi-stable implies that all such extensions are trivial. Thus $V=L_1\oplus L_2$. For later use, set d_i to be the degree of $L_i, i=1,2$. Then $d_1+d_2=d$ the degree of V, and

$$\#\mathrm{Aut}(V)=(q-1)^2\cdot q^{h^0(E,L_1\otimes L_2^{-1})}=(q-1)^2\cdot q^{d_1-d_2}.$$

Next we study the the contribution of degree 0 vector bundles of rank 2 which are not semi-stable. Note that the support of the summation should have non-vanishing h^0. Thus $V=L_1\oplus L_2$ where $L_1\in\mathrm{Pic}^{d_1}(E)$

with $d_1>0$. So the contributions of these bundles are given by

$$\begin{aligned}\zeta_{E,2}^{=0}(s)=&Z_{E,2}^{=0}(t)\\ =&\sum_{d=1}^{\infty}\sum_{L_1\in\mathrm{Pic}^d(E),L_2\in\mathrm{Pic}^{-d}(E)}\frac{q^{h^0(L_1)}-1}{(q-1)^2q^{h^0(L_1\otimes L_2^{\otimes -1})}}=\frac{N_1^2}{(q-1)^2}\cdot\sum_{d=1}^{\infty}\frac{q^d-1}{q^{2d}}\\ =&\frac{qN_1^2}{(q^2-1)(q-1)^2}.\end{aligned}$$

4.2.2. Degree >0

Now we consider all degree strictly positive rank 2 vector bundles which are not semi-stable. From above we see that $V=L_1\oplus L_2$ with $d_1>d_2$. Thus for $h^0(E,V)$, there are three cases:
(i) $d_2>0$, clearly then $h^0(E,V)=d$;
(ii) $d_2=0$. Here there are two subcases, namely, (a) if $L_2=\mathcal{O}_E$, then $h^0(E,V)=d_1+1$; (b) If $L_2\neq\mathcal{O}_E$, then $h^0(E,V)=d_1$;
(iii) $d_2<0$. Then $h^0(E,V)=d_1$.

Therefore, all in all the contribution of strictly positive degree rank 2 bundles which are not semi-stable to the zeta function $\zeta_{E,r}^{\mathrm{all}}(s)$ is given by

$$\zeta_{E,2}^{>0}(s)=Z_{E,2}^{>0}(t)=\Big(\sum_{(i)}+\sum_{(ii.a)}+\sum_{(ii.b)}+\sum_{(iii)}\Big)\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}t^d$$

where $\sum_{(*)}$ means the summation is taken for all vector bundles in case (*).

Hence, we have

$$\begin{aligned}\sum_{(i)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&N_1^2\cdot\sum_{d=1}^{\infty}\sum_{d_1+d_2=d,d_1>d_2>0}\frac{q^d-1}{(q-1)^2q^{d_1-d_2}}t^d,\\ \sum_{(ii.a)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&N_1\cdot\sum_{d=1}^{\infty}\frac{q^{d+1}-1}{(q-1)^2q^d}t^d,\\ \sum_{(ii.a)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&N_1(N_1-1)\cdot\sum_{d=1}^{\infty}\frac{q^d-1}{(q-1)^2q^d}t^d,\\ \sum_{(iii)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&N_1^2\cdot\sum_{d=1}^{\infty}\sum_{d_1+d_2=d,d_1>0>d_2}\frac{q^{d_1}-1}{(q-1)^2q^{d_1-d_2}}t^d.\end{aligned}$$

By a direct calculation, we find that

$$\begin{aligned}\sum_{(i)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&\frac{N_1^2t^3}{q-1}\cdot\frac{q^2+q+1+q^2t}{(1-t^2)(1-q^2t^2)(q-t)},\\ \sum_{(ii.a)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&\frac{N_1t}{q-1}\cdot\frac{q+1-t}{(q-t)(1-t)},\\ \sum_{(ii.a)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&\frac{N_1(N_1-1)}{q-1}\cdot\frac{t}{(q-t)(1-t)},\\ \sum_{(iii)}\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}=&\frac{N_1^2t}{(q-1)^2(q^2-1)}\cdot\frac{q^2+q-1-qt}{(1-t)(q-t)}.\end{aligned}$$

4.2.3. Degree <0

Finally we consider the contribution of bundles with strictly negative degree. First we have the following classification according to $h^0(E,V)$.

(i) $d_1 > 0 > d_2$. Then $h^0(E,V) = d_1$;

(ii) $d_1 = 0 > d_2$. Here two subcases. (a) $L_1 = \mathcal{O}_E$, then $h^0(E,V) = 1$; (b) $L_1 \neq \mathcal{O}_E$, then $h^0(V) = 0$;

(iii) $0 > d_1 > d_2$. Here $h^0(V) = 0$.

Thus note that the support of $h^0(E,V)$ is only on the cases (i) and (ii.a), we see that similarly as before, the contribution of strictly positive degree rank 2 bundles which are not semi-stable to the zeta function is given by

$$\zeta_{E,2}^{<0}(s) = Z_{E,2}^{<0}(t) = \Big(\sum_{(i)} + \sum_{(ii.a)}\Big)\frac{q^{h^0(V)}-1}{\#\mathrm{Aut}(V)}t^d.$$

Hence, we have

$$\begin{aligned}\zeta_{E,2}^{<0}(s) = Z_{E,2}^{<0}(t) =& N_1^2\sum_{d=-1}^{-\infty}\sum_{d_1>0>d_2, d_1+d_2=d}\frac{q^{d_1}-1}{(q-1)^2q^{d_1-d_2}}t^d + N_1\cdot\sum_{d=-1}^{-\subset nfty}\frac{q-1}{(q-1)^2q^{-d}}t^d\\ =&\frac{N_1^2}{(q-1)^2}\cdot\frac{q}{(qt-1)(q^2-1)} + \frac{N_1}{q-1}\cdot\frac{1}{qt-1}.\end{aligned}$$

I hope now the reader is fully convinced that our definition of non-abelian zeta function by using moduli space of semi-stable bundles is much better than that of others: Not only our semi-stable zeta functions have much neat structure, we also have well-behavior geometric and hence arithmetic spaces ready to use. In a certain sense, we think the picture of our non-abelian zeta function is quite similar to that the so-called new forms: Only after removing these not-semi-stable contributions, we can see the intrinsic beautiful structures.

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Appendix: Weierstrass Groups

1. Weierstrass Divisors

(1.1) Let M be a compact Riemann surface of genus $g \geq 2$. Denote its degree d Picard variety by $\mathrm{Pic}^d(M)$. Fix a Poincaré line bundle $\mathcal{P}_d$ on $M \times \mathrm{Pic}^d(M)$. (One checks easily that our constructions do not depend on a particular choice of Poincaré line bundle.) Let Θ be the theta divisor of $\mathrm{Pic}^{g-1}(M)$, i.e., the image of the natural map $M^{g-1} \to \mathrm{Pic}^{g-1}(M)$ defined by $(P_1, \ldots, P_{g-1}) \mapsto [\mathcal{O}_M(P_1 + \ldots + P_{g-1})]$. Here $[\cdot]$ denotes the class defined by $\cdot$. We will view the theta divisor as a pair $(\mathcal{O}_{\mathrm{Pic}^{g-1}(M)}(\Theta), \mathbf{1}_\Theta)$ with $\mathbf{1}_\Theta$ the defining section of Θ via the structure exact sequence $0 \to \mathcal{O}_{\mathrm{Pic}^{g-1}(M)} \to \mathcal{O}_{\mathrm{Pic}^{g-1}(M)}(\Theta)$.

Denote by p_i the i-th projection of $M \times M$ to M, $i = 1, 2$. Then for any degree $d = g - 1 + n$ line bundle on M, we get a line bundle $p_1^* L(-n\Delta)$ on $M \times M$ which has relative p_2-degree $g - 1$. Here, Δ denotes the diagonal divisor on $M \times M$. Hence, we get a classifying map $\phi_L : M \to \mathrm{Pic}^{g-1}(M)$ which makes the following diagram commute:

$$\begin{array}{ccc} M \times M & \rightarrow & M \times \mathrm{Pic}^{g-1}(M) \\ p_2 \downarrow & & \downarrow \pi \\ M & \xrightarrow{\phi_L} & \mathrm{Pic}^{g-1}(M). \end{array}$$

One checks that there are canonical isomorphisms

$$\lambda_\pi(\mathcal{P}^{g-1}) \simeq \mathcal{O}_{\mathrm{Pic}^{g-1}(M)}(-\Theta)$$

and

$$\lambda_{p_2}(p_1^* L(-n\Delta)) \simeq \phi_L^* \mathcal{O}_{\mathrm{Pic}^{g-1}(M)}(-\Theta).$$

Here, λ_π (resp. λ_{p_2}) denotes the Grothendieck-Mumford cohomology determinant with respect to π (resp. p_2). (See e.g., [L].)

Thus, $\phi_L^* \mathbf{1}_\Theta$ gives a canonical holomorphic section of the dual of the line bundle $\lambda_{p_2}(p_1^* L(-n\Delta))$, which in turn gives an effective divisor $W_L(M)$ on M, the so-called *Weierstrass divisor associated to* L.

Example. With the same notation as above, take $L = K_M^{\otimes m}$ with K_M the canonical line bundle of M and $m \in \mathbf{Z}$. Then we get an effective divisor $W_{K_M^{\otimes m}}(M)$ on M, which will be called the *m-th Weierstrass divisor* associated to M. For simplicity, denote $W_{K_M^{\otimes m}}(M)$ (resp. $\phi_{K_M^{\otimes m}}$) by $W_m(M)$ (resp. ϕ_m).

One checks easily that the degree of $W_m(M)$ is $g(g-1)^2(2m-1)^2$ and we have an isomorphism $\mathcal{O}_M(W_m(M)) \simeq K_M^{\otimes g(g-1)(2m-1)^2/2}$. Thus, in particular,

$$f_{m,n} := \frac{(\phi_m^* \mathbf{1}_\Theta)^{\otimes (2n-1)^2}}{(\phi_n^* \ \mathbf{1}_\Theta)^{\otimes (2m-1)^2}}$$

gives a canonical meromorphic function on M for all $m, n \in \mathbf{Z}$.

Remark. We may also assume that $m \in \frac{1}{2}\mathbf{Z}$. Furthermore, this construction has a relative version as well, for which we assume that $f : \mathcal{X} \to B$ is a semi-stable family of curves of genus $g \geq 2$. In that case, we get an effective divisor $(\mathcal{O}_\mathcal{X}(W_m(f)), \mathbf{1}_{W_m(f)})$ and canonical isomorphism

$$\begin{aligned} &(\mathcal{O}_\mathcal{X}(W_m(f)), \mathbf{1}_{W_m(f)}) \\ &\qquad \simeq (\mathcal{O}_\mathcal{X}(W_1(f)), \mathbf{1}_{W_1(f)})^{\otimes (2m-1)^2} \otimes (\mathcal{O}_\mathcal{X}(W_{\frac{1}{2}}(f)), \mathbf{1}_{W_{\frac{1}{2}}(f)})^{\otimes 4m(1-m)}. \end{aligned}$$

The proof may be given by using Deligne-Riemann-Roch theorem, which in general, implies that we have the following canonical isomorphism:

$$(\mathcal{O}_\mathcal{X}(W_L(f)), \mathbf{1}_{W_L(f)}) \otimes f^* \lambda_f(L) \simeq L^{\otimes n} \otimes K_f^{\otimes n(n-1)/2}.$$

(See e.g. [Bur].) To allow m be a half integer, we then should assume that f has a spin structure. Certainly, without using spin structure, a modified canonical isomorphism, valid for integers, can be given.

2. K-Groups

(2.1) Let M be a compact Riemann surface of genus $g \geq 2$. Then by the localization theorem, we get the following exact sequence for K-groups

$$K_2(M) \xrightarrow{\lambda} K_2(\mathbf{C}(M)) \xrightarrow{\coprod_{p\in M}\partial_p} \coprod_{p\in M} \mathbf{C}_p^*.$$

Note that the middle term may also be written as $K_2(\mathbf{C}(M\backslash S))$ for any finite subset S of M, we see that naturally by the theorem of Matsumoto, the Steinberg symbol $\{f_{m,n}, f_{m',n'}\}$ gives a well-defined element in $K_2(\mathbf{C}(M))$. Denote the subgroup generated by all $\{f_{m,n}, f_{m',n'}\}$ with $m,n,m',n' \in \mathbf{Z}_{>0}$ in $K_2(\mathbf{C}(M))$ as $\Sigma(M)$.

Definition. With the same notation as above, the *first Weierstrass group* $W_I(M)$ of M is defined to be the λ-pull-back of $\Sigma(M)$, i.e., the subgroup $\lambda^{-1}(\Sigma(M))$ of $K_2(M)$.

(2.2) For simplicity, now let C be a regular projective irreducible curve of genus $g \geq 2$ defined over $\mathbf{Q}$. Assume that C has a semi-stable regular module X over $\mathbf{Z}$ as well. Then we have a natural morphism $K_2(X) \xrightarrow{\phi} K_2(M)$. Here $M := C(\mathbf{C})$.

Conjecture I. *With the same notation as above,* $\phi\big(K_2(X)\big)_{\mathbf{Q}} = W_I(M)_{\mathbf{Q}}$.

3. Generalized Jacobians

(3.1) Let C be a projective, regular, irreducible curve. Then for any effective divisor D, one may canonically construct the so-called generalized Jacobian $J_D(C)$ together with a rational map $f_D : C \to J_D(C)$.

More precisely, let C_D be the group of classes of divisors prime to D modulo these which can be written as $\mathrm{div}(f)$. Let C_D^0 be the subgroup of C_D which consists of all elements of degree zero. For each p_i in the support of D, the invertible elements modulo those congruent to 1 (mod D) form an algebraic group R_{D,p_i} of dimension n_i, where n_i is the multiplicity of p_i in D. Let R_D be the product of these R_{D,p_i}. One checks easily that $\mathbf{G}_m$, the multiplicative group of constants naturally embeds into R_D. It is a classical result that we then have the short exact sequence

$$0 \to R_D/\mathbf{G}_m \to C_D^0 \to J \to 0$$

where J denotes the standard Jacobian of C. (See e.g., [S].) Denote $R_D/\mathbf{G}_m$ simply by $\mathbf{R}_D$.

Now the map f_D extends naturally to a bijection from C_D^0 to J_D. In this way the commutative algebraic group J_D becomes an extension as algebraic groups of the standard Jacobian by the group $\mathbf{R}_D$.

Example. Take the field of constants as $\mathbf{C}$ and $D = W_m(M)$, the m-th Weierstrass divisor of a compact Riemann surface M of genus $g \geq 2$. By (1.1), $W_m(M)$ is effective. So we get the associated generalized Jacobian $J_{W_m(M)}$. Denote it by $WJ_m(M)$ and call it the m-th *Weierstras-Jacobian* of M. For example, if $m = 0$, then $WJ_0(M) = J(M)$ is the standard Jacobian of M. Moreover, one knows that the dimension of $R_{W_m(M),p}$ is at most $g(g+1)/2$. For later use denote $\mathbf{R}_{W_m(M)}$ simply by $\mathbf{R}_m$.

(3.2) The above construction works on any base field as well. We leave the detail to the reader while point out that if the curve is defined over a field F, then its associated m-th Weierstrass divisor is rational over the same field as well. (Obviously, this is not true for the so-called Weierstrass points, which behavior in a rather random way.) As a consequence, by the construction of the generalized Jacobian, we see that the m-th Weierstrass-Jacobians are also defined over F. (See e.g., [S].)

4. Galois Cohomology Groups

(4.1) Let K be a perfect field, $\overline{K}$ be an algebraic closure of K and $G_{\overline{K}/K}$ be the Galois group of $\overline{K}$ over K. Then for any $G_{\overline{K}/K}$-module M, we have the Galois cohomology groups $H^0(G_{\overline{K}/K}, M)$ and $H^1(G_{\overline{K}/K}, M)$ such that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of $G_{\overline{K}/K}$-modules, then we get a natural long exact sequence

$$\begin{aligned}0 \to &H^0(G_{\overline{K}/K}, M_1) \to H^0(G_{\overline{K}/K}, M_2) \to H^0(G_{\overline{K}/K}, M_3) \\ &\to H^1(G_{\overline{K}/K}, M_1) \to H^1(G_{\overline{K}/K}, M_2) \to H^1(G_{\overline{K}/K}, M_3).\end{aligned}$$

Moreover, if G is a subgroup of $G_{\overline{K}/K}$ of finite index or a finite subgroup, then M is naturally a G-module. This leads a restriction map on cohomology res : $H^1(G_{\overline{K}/K}, M) \to H^1(G, M)$.

(4.2) Now let C be a projective, regular irreducible curve defined over a number field K. Then for each place p of K, fix an extension of p to $\overline{K}$, which then gives an embedding $\overline{K} \subset \overline{K}_p$ for the p-adic completion K_p of K and a decomposition group $G_p \subset G_{\overline{K}/K}$.

Now apply the construction in (3.1) to the short exact sequence

$$0 \to \mathbf{R}_m \to WJ_m(C) \to J(C) \to 0$$

over K. Then we have the following long exact sequence

$$\begin{aligned}0 \to &\mathbf{R}_m(K) \to WJ_m(K) \to J(K) \\ &\to H^1(G_{\overline{K}/K}, \mathbf{R}_m(K)) \to H^1(G_{\overline{K}/K}, WJ_m(K)) \xrightarrow{\psi} H^1(G_{\overline{K}/K}, J(K)).\end{aligned}$$

Similarly, for each place p of K, we have the following exact sequence

$$\begin{aligned}0 \to &\mathbf{R}_m(K_p) \to WJ_m(K_p) \to J(K_p) \\ &\to H^1(G_p, \mathbf{R}_m(K_p)) \to H^1(G_p, WJ_m(K_p)) \xrightarrow{\psi_p} H^1(G_p, J(K_p)).\end{aligned}$$

Now the natural inclusion $G_p \subset G_{\overline{K}/K}$ and $\overline{K} \subset \overline{K}_p$ give restriction maps on cohomology, so we arrive at a natural morphism

$$\Phi_m : \psi\Big(H^1(G_{\overline{K}/K}, WJ_m(K))\Big) \to \prod_{p \in M_K} \psi_p\Big(H^1(G_p, WJ_m(K_p))\Big).$$

Here M_K denotes the set of all places over K.

Definition. With the same notation as above, *the second Weierstrass group* $W_{II}(C)$ of C is defined to be the subgroup of $H^1(G_{\overline{K}/K}, J(C)(K))$ generated by all $\operatorname{Ker}\Phi_m$, the kernel of Φ_m, i.e., $W_{II}(C) := \langle \operatorname{Ker}\Phi_m : m \in \mathbf{Z}_{>0}\rangle_{\mathbf{Z}}$.

Conjecture II. *With the same notation as above, the second Weierstrass group $W_{II}(C)$ is finite.*

5. Deligne-Beilinson Cohomology

(5.1) Let C be a projective regular curve of genus g. Let P be a finite set of C. For simplicity, assume that all of them are defined over $\mathbf{R}$. Then we have the associated Deligne-Belinsion cohomology group $H^1_{\mathcal{D}}(C\backslash P, \mathbf{R}(1))$ which leads to the following short exact sequence:

$$0 \to \mathbf{R} \to H^1_{\mathcal{D}}(C\backslash P, \mathbf{R}(1)) \xrightarrow{\text{div}} \mathbf{R}[P]^0 \to 0$$

where $\mathbf{R}[P]^0$ denotes (the group of degree zero divisors with support on $P)_{\mathbf{R}}$.

The standard cup product on Deligne-Beilinson cohomology leads to a well-defined map:

$$\cup : H^1_{\mathcal{D}}(C\backslash P, \mathbf{R}(1)) \times H^1_{\mathcal{D}}(C\backslash P, \mathbf{R}(1)) \to H^2_{\mathcal{D}}(C\backslash P, \mathbf{R}(2)).$$

Furthermore, by Hodge theory, there is a canonical short exact sequence

$$0 \to H^1(C\backslash P, \mathbf{R}(1)) \cap F^1(C\backslash P) \to H^2_{\mathcal{D}}(C\backslash P, \mathbf{R}(2)) \xrightarrow{p_{\mathcal{D}}} H^2_{\mathcal{D}}(C, \mathbf{R}(2)) \to 0$$

where F^1 denotes the F^1-term of the Hodge filtration on $H^1(C\backslash P, \mathbf{C})$.

All this then leads to a well-defined morphism

$$[\cdot,\cdot]_{\mathcal{D}} : \wedge^2 \mathbf{R}[P]^0 \to H^2_{\mathcal{D}}(C, \mathbf{R}(2)) = H^1(C, \mathbf{R}(1))$$

which make the associated diagram coming from the above two short exact sequences commute. (See e.g. [Bei].)

(5.2) Now applying the above construction with P being the union of the supports of W_1, W_m and W_n for $m, n > 0$. Thus for fixed m, n, in $\mathbf{R}[P]^0$, we get two elements $\mathrm{div}(f_{1,m})$ and $\mathrm{div}(f_{1,n})$. This then gives $[\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}} \in H^1(X, \mathbf{R}(1))$.

Lemma. *For any holomorphic differential 1-form ω on C, we have*

$$\langle [\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}}, \omega\rangle := -\frac{1}{2\pi\sqrt{-1}} \int [\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}} \wedge \bar{\omega}$$
$$= -\frac{1}{2\pi\sqrt{-1}} \int g(\mathrm{div}(f_{1,m}), z) dg(\mathrm{div}(f_{1,n}), z) \wedge \bar{\omega},$$

Here $g(D, z)$ denotes the Green's function of D with respect to any fixed normalized (possibly singular) volume form of quasi-hyperbolic type.

Proof. A simple argument by using the Stokes formula.

(4.3) With exactly the same notation as in (4.2), then in $H^1(X, \mathbf{R}(1))$ we get a collection of elements $[\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}}$ for $m, n \in \mathbf{Z}_{>0}$.

Definition. With the same notation as above, assume that C is defined over $\mathbf{Z}$. Define the *–first quasi-Weierstrass group $W'_{-I}(C)$* of C to be the subgroup of $H^1(X, \mathbf{R}(1))$ generated by $[\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}}$ for all $m, n \in \mathbf{Z}_{>0}$ and call $W'_{-I}(C)_{\mathbf{Q}}$ the *–first Weierstrass group $W_{-I}(C)$* of C. That is to say, $W_{-I}(C) := \langle [\mathrm{div}(f_{1,m}), \mathrm{div}(f_{1,n})]_{\mathcal{D}} : m, n \in \mathbf{Z}_{>0}\rangle_{\mathbf{Q}}$.

Conjecture III. *With the same notation as above, $W_{-1}(C)_{\mathbf{R}}$ is the full space, i.e. equals to $H^1(X, \mathbf{R}(1))$.*

That is to say, we believe Weierstrass divisors will give a new rational structure for $H^1(X, \mathbf{R}(1))$. Furthermore, we believe that the corresponding regulator will give the leading coefficient of the L-function of C at $s = 0$, up to rationals.

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